

# Semiclassical Dynamics of the Jaynes-Cummings Model

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## Abstract

The semiclassical approximation of coherent state path integrals is employed to study the dynamics of the Jaynes-Cummings model. Decomposing the Hilbert space into subspaces of given excitation quanta above the ground state, the semiclassical propagator is shown to describe the exact quantum dynamics of the model. We also present a semiclassical approximation that does not exploit the special properties of the Jaynes-Cummings Hamiltonian and can be extended to more general situations. In this approach the contribution of the dominant semiclassical paths and the relevant fluctuations about them are evaluated. This theory leads to an accurate description of spontaneous emission going beyond the usual classical field approximation.

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## I. INTRODUCTION

Since the sixties of the last century the Jaynes-Cummings model [1] is frequently considered as a simple model to describe a two-level atomic system interacting with an electromagnetic field in a cavity; for recent reviews see [2,3]. Apart from its relevance to quantum optics, in particular laser theory, this integrable quantum model also allows to test approximative methods by comparing them with the exact result. In particular, the “semiclassical” theory has attracted considerable attention where the bosonic field mode is represented by classical c-numbers while the two-level atomic system is represented as a quantum spin- $\frac{1}{2}$  [4]- [6]. In this approximation the Heisenberg equations of motion [7] are replaced by linear operator equations for the spin variables and an amplitude equation for the electromagnetic field which is driven by the expectation values of the spin operators. Taking the expectation value of the Heisenberg equations for the spin variables, the optical Bloch equations emerge which describe the dynamics of a classical Bloch vector on the two-sphere [8,9]. It is well known that this “semiclassical” theory provides results that are equivalent to a full quantum mechanical treatment if the mean number of bosons is very large and fluctuations in the boson number can be neglected [10].

While this conventional semiclassical approach treats the cavity field just classically, we attempt at a semiclassical theory treating both the atomic and electromagnetic subsystems on an equal footing. Starting from the full quantum model we focus on the most probable paths of the system within the path integral representation and relevant fluctuations about them. Within the scope of (spin) coherent state path integrals we obtain a semiclassical approximation going beyond the classical field approximation. For instance, the approach yields an accurate description of spontaneous emission.

The paper is organized as follows. In Sec.II we first solve the Jaynes-Cummings model exactly with spin coherent state path integrals in a subspace with fixed excitation quanta above the ground state. Then, in Sec.III, we examine a semiclassical description which does not rely on these subspaces and can thus be extended to more complicated Hamiltonians. With coherent state path integrals the leading order of the propagator is determined by solving the Euler-Lagrange equations for the classical path. In Sec.IV we consider contributions from fluctuations about the dominant path and show that they lead to a decay of the excited two-level system by spontaneous emission.

## II. THE JAYNES-CUMMINGS MODEL

The Jaynes-Cummings model is characterized by the Hamiltonian

$$H = a^\dagger a + (1 + \Delta)S_z + \lambda(aS_+ + a^\dagger S_-), \quad (1)$$

where  $a$  is the canonical annihilation operator of a bosonic field mode with frequency  $\omega$  and  $S_\pm = S_x \pm iS_y$ ,  $S_z$  are operators of a spin- $\frac{1}{2}$  describing two levels of an atomic system with energy difference  $\hbar\omega_o$ . There are two dimensionless parameters, the detuning  $\Delta = (\omega_o - \omega)/\omega$  and the coupling strength  $\lambda = g/\omega$ . We use units with  $\omega = 1$  and  $\hbar = 1$ . It is well known that the Jaynes-Cummings model allows apart from  $H$  for another time independent operator [7]

$$N = a^\dagger a + S_z, \quad (2)$$

which measures the number of excitation quanta in the system. Hence, the time evolution operator is of the form

$$U(T) = e^{-iHT} = e^{-iNT} e^{-iCT}, \quad (3)$$

where  $C = H - N$ . Representing the spin operators in the eigenbasis of  $S_z$  formed by the eigenvectors  $|\uparrow\rangle$  and  $|\downarrow\rangle$ , the first factor in Eq.(3) may be written as

$$e^{-iNT} = e^{-ia^\dagger a T} \left( e^{-\frac{i}{2}T} |\uparrow\rangle\langle\uparrow| + e^{+\frac{i}{2}T} |\downarrow\rangle\langle\downarrow| \right). \quad (4)$$

Introducing further the eigenkets of  $a^\dagger a$ , invariant subspaces are distinguished. In particular the kets  $|\uparrow n-1\rangle \equiv |\uparrow\rangle|n-1\rangle$  and  $|\downarrow n\rangle \equiv |\downarrow\rangle|n\rangle$  span the subspace with  $N = (n - \frac{1}{2})$ . In this subspace the time independent operator  $C$  generates  $SU(2)$  dynamics. This can be seen explicitly by introducing the operators

$$\begin{aligned} J_x &= \frac{1}{2} \left( |\uparrow n-1\rangle\langle\downarrow n| + |\downarrow n\rangle\langle\uparrow n-1| \right) \\ J_y &= \frac{i}{2} \left( -|\uparrow n-1\rangle\langle\downarrow n| + |\downarrow n\rangle\langle\uparrow n-1| \right) \\ J_z &= \frac{1}{2} \left( |\uparrow n-1\rangle\langle\uparrow n-1| - |\downarrow n\rangle\langle\downarrow n| \right), \end{aligned} \quad (5)$$

describing the angular momentum of a spin- $\frac{1}{2}$ . In terms of these spin operators we have

$$C = 2\lambda\sqrt{n}J_x + \Delta J_z, \quad (6)$$

and we see that in this subspace  $C$  gives indeed rise to pure  $SU(2)$  dynamics. Accordingly, the propagator may be worked out exactly by a semiclassical approach with path integrals in the spin coherent state representation

$$|\vartheta \varphi\rangle = e^{-i\varphi J_z} e^{-i\vartheta J_y} |\uparrow n-1\rangle. \quad (7)$$

Following the lines of [11], we write the spin coherent propagator as a regularized path integral

$$\langle \vartheta'' \varphi'' | e^{-iCT} | \vartheta' \varphi' \rangle = \lim_{\nu \rightarrow \infty} \int d\mu \exp \{ i S[\vartheta(t), \varphi(t)] \}, \quad (8)$$

with the action

$$S[\vartheta(t), \varphi(t)] = \int_0^T dt \left[ \frac{1}{2} \cos(\vartheta) \dot{\varphi} - C(\vartheta, \varphi) \right]. \quad (9)$$

Here the operator  $C$  is represented as

$$\begin{aligned} C(\vartheta, \varphi) &= \langle \vartheta \varphi | C | \vartheta \varphi \rangle. \\ &= \lambda\sqrt{n} \sin(\vartheta) \cos(\varphi) + \frac{\Delta}{2} \cos(\vartheta). \end{aligned} \quad (10)$$

The spherical Wiener measure [12,13]

$$d\mu = M \prod_{t=0}^T d\cos(\vartheta(t)) d\varphi(t) \exp \left\{ -\frac{1}{4\nu} \int_0^T dt \left[ \dot{\vartheta}^2 + \sin^2(\vartheta) \dot{\varphi}^2 \right] \right\}, \quad (11)$$

enforces continuous Brownian motion paths on the sphere ( $M$  is a normalization factor). This measure gives rise to a regularization dependent action

$$S_\nu[\vartheta(t), \varphi(t)] = \int_0^T dt \left\{ \frac{i}{4\nu} \left[ \dot{\vartheta}^2 + \sin^2(\vartheta) \dot{\varphi}^2 \right] + \frac{1}{2} \cos(\vartheta) \dot{\varphi} - C(\vartheta, \varphi) \right\}. \quad (12)$$

Now, in the semiclassical expansion, we separate the paths  $\cos(\vartheta) = \cos(\vartheta_{cl}) + x/\sqrt{s}$  and  $\varphi = \varphi_{cl} + y/\sqrt{s}$  in their classical parts and fluctuations around them. The formal limit of large spin  $s \rightarrow \infty$  expresses the classical limit. To lowest order, in the Dominant Path Approximation (DOPA), the semiclassical expansion gives

$$e^{iS_{cl}} = \exp \left\{ -i \int_0^T dt C(\bar{\vartheta}''(t), \bar{\varphi}''(t)) \right\} \langle \vartheta'' \varphi'' | \vartheta' \varphi' \rangle, \quad (13)$$

where  $(\bar{\vartheta}', \bar{\varphi}')$  and  $(\bar{\vartheta}'', \bar{\varphi}'')$ , respectively, describe the starting point and endpoint of the classical trajectory  $(\bar{\vartheta}(t), \bar{\varphi}(t))$ ,  $0 \leq t \leq T$ . For convenience let us introduce the complex variables

$$\begin{aligned} \zeta &= \tan \left( \frac{\bar{\vartheta}}{2} \right) e^{i\bar{\varphi}} \\ \eta &= \tan \left( \frac{\bar{\vartheta}}{2} \right) e^{-i\bar{\varphi}}. \end{aligned} \quad (14)$$

Then, the dominant path is determined by

$$\begin{aligned} \dot{\zeta} &= -i\lambda\sqrt{n}(1 - \zeta^2) + i\Delta\zeta \\ \dot{\eta} &= i\lambda\sqrt{n}(1 - \eta^2) - i\Delta\eta, \end{aligned} \quad (15)$$

with boundary conditions  $\zeta(0) = \zeta'$  and  $\eta(T) = \eta''$ . Hence, the endpoint of the classical trajectory obeys

$$\begin{aligned} \zeta(T) &= \frac{2\Omega_n \zeta' \cos(\Omega_n T) + i[\Delta\zeta' - \lambda\sqrt{n}] \sin(\Omega_n T)}{2\Omega_n \zeta' \cos(\Omega_n T) - i[\lambda\sqrt{n}\zeta' + \Delta] \sin(\Omega_n T)} \\ \eta(T) &= \eta'', \end{aligned} \quad (16)$$

with the Rabi frequency

$$\Omega_n = \sqrt{\lambda^2 n + \frac{\Delta^2}{4}}. \quad (17)$$

In terms of the complex variables (14) we get

$$\begin{aligned} C(\zeta(T), \eta'') &= \lambda\sqrt{n} \frac{\zeta(T) + \eta}{1 + \zeta(T)\eta''} + \frac{\Delta}{2} \frac{1 - \zeta(T)\eta''}{1 + \zeta(T)\eta''} \\ &= i \frac{d}{dT} \log \left\{ (1 + \zeta'\eta'') \cos(\Omega_n T) \right. \\ &\quad \left. - \frac{i}{\Omega_n} [\lambda\sqrt{n}(\zeta' + \eta'') + \frac{\Delta}{2}(1 - \zeta'\eta'')] \sin(\Omega_n T) \right\}. \end{aligned} \quad (18)$$

Now the integral in Eq.(13) is readily solved and the propagator in the DOPA takes the form

$$e^{iS_{cl}} = a(T) \cos\left(\frac{\vartheta''}{2}\right) \cos\left(\frac{\vartheta'}{2}\right) e^{\frac{i}{2}(\varphi'' - \varphi')} + b(T) \cos\left(\frac{\vartheta''}{2}\right) \sin\left(\frac{\vartheta'}{2}\right) e^{\frac{i}{2}(\varphi'' + \varphi')} \\ - b^*(T) \sin\left(\frac{\vartheta''}{2}\right) \cos\left(\frac{\vartheta'}{2}\right) e^{-\frac{i}{2}(\varphi'' + \varphi')} + a^*(T) \sin\left(\frac{\vartheta''}{2}\right) \sin\left(\frac{\vartheta'}{2}\right) e^{-\frac{i}{2}(\varphi'' - \varphi')}, \quad (19)$$

where

$$a(T) = \cos(\Omega_n T) - i \frac{\Delta}{2\Omega_n} \sin(\Omega_n T) \\ b(T) = -i \frac{\lambda \sqrt{n}}{\Omega_n} \sin(\Omega_n T). \quad (20)$$

As discussed elsewhere [11] for pure  $SU(2)$  dynamics the DOPA is exact and Eqs.(19),(20) give indeed the exact propagator [3].

In more general situations, such as for the case without rotating wave approximation [14,15], the system cannot be separated into invariant subspaces. Therefore it would be interesting to consider a semiclassical expansion that does not rely on the  $SU(2)$  generators (5).

### III. SEMICLASSICAL DYNAMICS WITH COHERENT STATE PATH INTEGRALS

In order to formulate a general semiclassical theory for a coupled spin boson problem we make use of product coherent states

$$|\vartheta \varphi p q\rangle = e^{-i\varphi S_z} e^{-i\vartheta S_y} e^{i(pQ - qP)} |\uparrow\rangle |0\rangle. \quad (21)$$

These states are generated by momentum and space translations of the normalized vacuum state  $|0\rangle$  and  $SU(2)$  rotations of the  $S_z$  eigenstate  $|\uparrow\rangle$ . Again, the semiclassical approximation is based on the coherent state path integral representation. We express the propagator as

$$\langle \vartheta'' \varphi'' p'' q'' | U(t) | \vartheta' \varphi' p' q' \rangle = \lim_{\nu_a, \nu_b \rightarrow \infty} \int d\mu_a d\mu_b \exp \{ i S[p(t), q(t), \vartheta(t), \varphi(t)] \}, \quad (22)$$

with the action

$$S[p(t), q(t), \vartheta(t), \varphi(t)] = \int_0^T dt \left[ \frac{1}{2} (p\dot{q} - \dot{p}q) + \frac{1}{2} \cos(\vartheta) \dot{\varphi} - H(\vartheta, \varphi, p, q) \right]. \quad (23)$$

For the Jaynes-Cummings model the Hamiltonian takes the form

$$H(\vartheta, \varphi, p, q) = \langle \vartheta \varphi p q | H | \vartheta \varphi p q \rangle \\ = \frac{1}{2} (p^2 + q^2) + \frac{1 + \Delta}{2} \cos(\vartheta) + \frac{\lambda}{2\sqrt{2}} \left[ \sin(\vartheta) e^{i\varphi} (q + ip) + \sin(\vartheta) e^{-i\varphi} (q - ip) \right]. \quad (24)$$

Here, the canonical coherent state path integral is regularized by the flat Wiener measure [12,13]

$$d\mu_a = M_a \prod_{t=0}^T dp(t) dq(t) \exp \left\{ -\frac{1}{2\nu_a} \int_0^T dt [\dot{q}^2 + \dot{p}^2] \right\}, \quad (25)$$

while the spin paths are again regularized by the spherical Wiener measure

$$d\mu_b = M_b \prod_{t=0}^T d\cos(\vartheta(t)) d\varphi(t) \exp \left\{ -\frac{1}{4\nu_b} \int_0^T dt [\dot{\vartheta}^2 + \sin^2(\vartheta) \dot{\varphi}^2] \right\}. \quad (26)$$

These measures give rise to the regularization dependent action

$$S_{\nu_a, \nu_b}[\vartheta(t), \varphi(t), p(t), q(t)] = \int_0^T dt \left\{ \frac{i}{2\nu_a} [\dot{q}^2 + \dot{p}^2] + \frac{i}{4\nu_b} [\dot{\vartheta}^2 + \sin^2(\vartheta) \dot{\varphi}^2] \right. \\ \left. + \frac{1}{2}(p\dot{q} - \dot{p}q) + \frac{1}{2} \cos(\vartheta) \dot{\varphi} - H(\vartheta, \varphi, p, q) \right\}. \quad (27)$$

In the semiclassical expansion we split the paths  $p = p_{cl} + x_a$ ,  $q = q_{cl} + y_a$ ,  $\cos(\vartheta) = \cos(\vartheta_{cl}) + x_b/\sqrt{s}$  and  $\varphi = \varphi_{cl} + y_b/\sqrt{s}$  in their classical parts and fluctuations around them. Restricting ourselves to the DOPA, we obtain the propagator

$$e^{iS_{cl}} = \sqrt{\frac{\sin(\vartheta') \sin(\vartheta'')}{\sin(\bar{\vartheta}') \sin(\bar{\vartheta}'')}} \exp \left\{ -\frac{1}{2} [q'' \bar{p}'' - \bar{q}'' p'' + q' p' - q' \bar{p}'] \right\} \\ \times \exp \left\{ i \int_0^T dt \left[ \frac{1}{2} \cos(\bar{\vartheta}) \dot{\bar{\varphi}} + \frac{1}{2} (\bar{p} \dot{\bar{q}} - \dot{\bar{p}} \bar{q}) - H(\bar{\vartheta}, \bar{\varphi}, \bar{p}, \bar{q}) \right] \right\}. \quad (28)$$

While for  $\lambda = 0$  this approximation yields the exact propagator, this property is lost for the interacting system. Introducing the complex variables [16,17]

$$\alpha = \frac{1}{\sqrt{2}}(\bar{q} + i\bar{p}) \\ \beta = \frac{1}{\sqrt{2}}(\bar{q} - i\bar{p}) \\ \zeta = \tan \left( \frac{\bar{\vartheta}}{2} \right) e^{i\bar{\varphi}} \\ \eta = \tan \left( \frac{\bar{\vartheta}}{2} \right) e^{-i\bar{\varphi}}, \quad (29)$$

Eq.(28) may be expressed as

$$e^{iS_{cl}} = \exp \left\{ -\frac{1}{2} [|\alpha'|^2 + |\beta''|^2 - \alpha(T)\beta'' - \alpha'\beta(0)] \right\} \sqrt{\frac{(1 + \zeta'\eta(0))(1 + \zeta(T)\eta'')}{(1 + \zeta'\eta')(1 + \zeta''\eta'')}} \\ \times \left( \frac{\zeta''\eta'}{\zeta'\eta''} \right)^{\frac{1}{4}} \exp \left\{ i \int_0^T dt \left[ \frac{i}{2} (\dot{\alpha}\beta - \alpha\dot{\beta}) + \frac{i}{2} \frac{\dot{\zeta}\eta - \zeta\dot{\eta}}{1 + \zeta\eta} - H(\alpha, \beta, \zeta, \eta) \right] \right\}, \quad (30)$$

where the Hamiltonian (24) reads

$$H(\alpha, \beta, \zeta, \eta) = \alpha\beta + \frac{1 + \Delta}{2} \frac{1 - \zeta\eta}{1 + \zeta\eta} + \lambda \frac{\alpha\zeta + \beta\eta}{1 + \zeta\eta}. \quad (31)$$

Now, the DOPA propagator is determined by the dominant path obeying the classical equations of motion

$$\begin{aligned} \dot{\alpha} &= -i \left[ \alpha + \lambda \frac{\eta}{1 + \zeta\eta} \right] \\ \dot{\beta} &= i \left[ \beta + \lambda \frac{\zeta}{1 + \zeta\eta} \right] \\ \dot{\zeta} &= i \left[ (1 + \Delta)\zeta - \lambda(\beta - \alpha\zeta^2) \right] \\ \dot{\eta} &= -i \left[ (1 + \Delta)\eta - \lambda(\alpha - \beta\eta^2) \right], \end{aligned} \quad (32)$$

with the boundary conditions

$$\begin{aligned} \alpha(0) &= \frac{1}{\sqrt{2}}(q' + i p') \\ \beta(T) &= \frac{1}{\sqrt{2}}(q'' - i p'') \\ \zeta(0) &= \tan\left(\frac{\vartheta'}{2}\right) e^{i\varphi'} \\ \eta(T) &= \tan\left(\frac{\vartheta''}{2}\right) e^{-i\varphi''}. \end{aligned} \quad (33)$$

The system of differential equations (32) gives rise to a Hamiltonian vector field. Extending results in [17,18], one sees that this Hamiltonian dynamics is identical to the classical mechanics of a spin on the two-sphere coupled to the phase space degrees of freedom of a one dimensional harmonic oscillator. Since the covariant divergence of the Hamiltonian vector field vanishes, this dynamical system is conservative and no attractor can occur. The coupled differential equations (32) with conditions (33) express a nonlinear boundary value problem. We can find a solution exploiting the invariance of the action (23) under phase transformations

$$\begin{aligned} \zeta &\rightarrow \zeta e^{i\Lambda} \\ \eta &\rightarrow \eta e^{-i\Lambda} \\ \alpha &\rightarrow \alpha e^{-i\Lambda} \\ \beta &\rightarrow \beta e^{i\Lambda}. \end{aligned} \quad (34)$$

The corresponding integral of motion is

$$N(\alpha, \beta, \zeta, \eta) = \alpha\beta + \frac{1}{2} \frac{1 - \zeta\eta}{1 + \zeta\eta}. \quad (35)$$

Therefore the Hamiltonian dynamical system becomes integrable by the theorem of Liouville-Arnold [19]. Particularly, by setting  $u = (1 - \zeta\eta)/(1 + \zeta\eta)$ , we reduce the system to a one-dimensional problem of the form

$$\frac{1}{2}\dot{u}^2 + V(u) = 0, \quad (36)$$

with the cubic potential

$$V(u) = \lambda^2(u^3 + a_2u^2 + a_1u + a_0). \quad (37)$$

The coefficients read

$$\begin{aligned} a_0 &= -2N + 2\frac{C^2}{\lambda^2} \\ a_1 &= 1 + 2\frac{\Delta C}{\lambda^2} \\ a_2 &= 2N + \frac{\Delta^2}{2\lambda^2}, \end{aligned} \quad (38)$$

where

$$C(\alpha, \beta, \zeta, \eta) = \lambda \frac{\alpha\zeta + \beta\eta}{1 + \zeta\eta} + \frac{\Delta}{2} \frac{1 - \zeta\eta}{1 + \zeta\eta}. \quad (39)$$

Although the potential  $V(u)$  is time independent, the boundary values (33) enforce the coefficients in Eq.(38) to depend on the end time  $T$ , and the form of  $V(u)$  changes with  $T$ . Next we set  $v = u + a_2/3$  and rewrite Eq.(36) as

$$\frac{2}{\lambda^2}\dot{v}^2 = 4v^3 - g_2v - g_3. \quad (40)$$

This is just the differential equation solved by the Weierstrass elliptic function [20]  $\wp(\frac{\lambda}{\sqrt{2}}t; g_2; g_3)$  with the invariants

$$\begin{aligned} g_2 &= -4 \left( a_1 - \frac{1}{3}a_2^2 \right) \\ g_3 &= -\frac{4}{3} \left( \frac{2}{9}a_2^3 - a_1a_2 \right) - 4a_0. \end{aligned} \quad (41)$$

In the following we suppress these invariants in the list of arguments of the function  $\wp$ . Now, the solution of Eq.(36) becomes

$$u(t) = -\frac{a_2}{3} + \wp \left( A_1 + \frac{\lambda}{\sqrt{2}}t \right), \quad (42)$$

where

$$A_1 = \wp^{-1} \left( \frac{a_2}{3} + \frac{1 - \zeta'\eta(0)}{1 + \zeta'\eta(0)} \right) \quad (43)$$

is determined by the inverse Weierstrass function  $\wp^{-1}$ . Making use of the solution (42), the equations of motion lead to elliptic integrals which can be solved in terms of the Weierstrass elliptic functions  $\wp$ ,  $\zeta_w$  and  $\sigma_w$  [20]. After some algebra one finds for the field coordinates

$$\begin{aligned}
\alpha(t) &= \alpha' \left[ \frac{1}{2\alpha'\beta(0)} \frac{\sigma_w(A_2 + A_1)}{\sigma_w(A_2 - A_1)} \right]^{1/2} \\
&\times \exp\left\{ \frac{\lambda}{\sqrt{2}} \zeta_w(A_2) t \right\} \exp\left\{ -i \left( 1 + \frac{\Delta}{2} \right) t \right\} \\
&\times \left[ \left( 2N + \frac{a_2}{3} - \wp(A_1 + \frac{\lambda}{\sqrt{2}}t) \right) \frac{\sigma_w(A_2 - A_1 + \frac{\lambda}{\sqrt{2}}t)}{\sigma_w(A_2 + A_1 + \frac{\lambda}{\sqrt{2}}t)} \right]^{1/2},
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
\beta(t) &= \beta'' \left[ \frac{1}{2\alpha(T)\beta''} \frac{\sigma_w(A_2 - A_1 - \frac{\lambda}{\sqrt{2}}T)}{\sigma_w(A_2 + A_1 + \frac{\lambda}{\sqrt{2}}T)} \right]^{1/2} \\
&\times \exp\left\{ \frac{\lambda}{\sqrt{2}} \zeta_w(A_2)(T - t) \right\} \exp\left\{ -i \left( 1 + \frac{\Delta}{2} \right) (T - t) \right\} \\
&\times \left[ \left( 2N + \frac{a_2}{3} - \wp(A_1 + \frac{\lambda}{\sqrt{2}}t) \right) \frac{\sigma_w(A_2 + A_1 - \frac{\lambda}{\sqrt{2}}t)}{\sigma_w(A_2 - A_1 - \frac{\lambda}{\sqrt{2}}t)} \right]^{1/2},
\end{aligned} \tag{45}$$

where

$$A_2 = \wp^{-1} \left( \frac{a_2}{3} + 2N \right). \tag{46}$$

The spin variables are found to read

$$\begin{aligned}
\zeta(t) &= \zeta' \left[ \frac{1}{\zeta'\eta(0)} \frac{\sigma_w(A_3 + A_1)}{\sigma_w(A_3 - A_1)} \frac{\sigma_w(A_4 + A_1)}{\sigma_w(A_4 - A_1)} \right]^{1/2} \\
&\times \exp\left\{ \frac{\lambda}{\sqrt{2}} [\zeta_w(A_3) + \zeta_w(A_4)] t \right\} \exp\{it\} \\
&\times \left[ \frac{1 + \frac{a_2}{3} - \wp(A_1 + \frac{\lambda}{\sqrt{2}}t)}{1 - \frac{a_2}{3} + \wp(A_1 + \frac{\lambda}{\sqrt{2}}t)} \frac{\sigma_w(A_3 - A_1 + \frac{\lambda}{\sqrt{2}}t)}{\sigma_w(A_3 + A_1 + \frac{\lambda}{\sqrt{2}}t)} \frac{\sigma_w(A_4 - A_1 + \frac{\lambda}{\sqrt{2}}t)}{\sigma_w(A_4 + A_1 + \frac{\lambda}{\sqrt{2}}t)} \right]^{1/2}
\end{aligned} \tag{47}$$

and

$$\begin{aligned}
\eta(t) &= \eta'' \left[ \frac{1}{\zeta(T)\eta''} \frac{\sigma_w(A_3 + A_1 + \frac{\lambda}{\sqrt{2}}T)}{\sigma_w(A_3 - A_1 + \frac{\lambda}{\sqrt{2}}T)} \frac{\sigma_w(A_4 + A_1 + \frac{\lambda}{\sqrt{2}}T)}{\sigma_w(A_4 - A_1 + \frac{\lambda}{\sqrt{2}}T)} \right]^{1/2} \\
&\times \exp\left\{ \frac{\lambda}{\sqrt{2}} [\zeta_w(A_3) + \zeta_w(A_4)](T - t) \right\} \exp\{i(T - t)\} \\
&\times \left[ \frac{1 + \frac{a_2}{3} - \wp(A_1 + \frac{\lambda}{\sqrt{2}}t)}{1 - \frac{a_2}{3} + \wp(A_1 + \frac{\lambda}{\sqrt{2}}t)} \frac{\sigma_w(A_3 - A_1 + \frac{\lambda}{\sqrt{2}}t)}{\sigma_w(A_3 + A_1 + \frac{\lambda}{\sqrt{2}}t)} \frac{\sigma_w(A_4 - A_1 + \frac{\lambda}{\sqrt{2}}t)}{\sigma_w(A_4 + A_1 + \frac{\lambda}{\sqrt{2}}t)} \right]^{1/2}
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
A_3 &= \wp^{-1} \left( \frac{a_2}{3} - 1 \right) \\
A_4 &= \wp^{-1} \left( \frac{a_2}{3} + 1 \right).
\end{aligned} \tag{49}$$

The solutions (44)-(49) give the dominant path in terms of the known initial values  $\alpha(0) = \alpha'$ ,  $\zeta(0) = \zeta'$  and final values  $\beta(T) = \beta''$ ,  $\eta(T) = \eta''$  and as implicit functions of the unknown initial values  $\beta(0)$ ,  $\eta(0)$  and final values  $\alpha(T)$ ,  $\zeta(T)$ . Two of these unknowns have to be determined numerically. For instance, from Eqs.(44) and (47) we obtain two transcendental equations for  $\alpha(T)$  and  $\zeta(T)$  that can be solved by a root search procedure. Then, the two other unknowns can be found from the two constants  $C$  and  $N$ .

Having determined the semiclassical trajectory, we may insert the result into Eq.(30) and determine the DOPA-Propagator. Since this propagator obeys a semiclassical Schrödinger equation [see Appendix A], an alternative representation of the propagator reads

$$e^{iS_{cl}} = \exp \left\{ -i \int_0^T dt H(\alpha(t), \beta'', \zeta(t), \eta'') \right\} \langle \vartheta'' \varphi'' | \vartheta' \varphi' \rangle \langle p'' q'' | p' q' \rangle. \quad (50)$$

With this representation the DOPA propagator is just determined by the the endpoint of the classical path.

Although the dynamical system (32) is conservative, it gives rise to stationary states. These are the fix points  $(\alpha, \beta, \zeta_N, \eta_N) = (0, 0, 0, 0)$  and  $(\alpha, \beta, \rho_S, \sigma_S) = (0, 0, 0, 0)$ , where  $\rho = 1/\zeta$  and  $\sigma = 1/\eta$ . These points correspond to the states  $|\uparrow 0\rangle$  and  $|\downarrow 0\rangle$  referred to as north pole and south pole, henceforth. For a linear stability analysis we just have to linearize the spin terms since the equations of motion (32) are already linear in the oscillator variables. Expanding about  $(\zeta_N, \eta_N)$  we find

$$\frac{d}{dt} \begin{pmatrix} \delta\zeta \\ \beta \\ \delta\eta \\ \alpha \end{pmatrix} = i \begin{pmatrix} 1 + \Delta & -\lambda & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ 0 & 0 & -1 - \Delta & \lambda \\ 0 & 0 & -\lambda & -1 \end{pmatrix} \begin{pmatrix} \delta\zeta \\ \beta \\ \delta\eta \\ \alpha \end{pmatrix}, \quad (51)$$

and two invariant subspaces in the variables  $(\delta\zeta, \beta)$  and  $(\delta\eta, \alpha)$  appear. The solution satisfying the boundary conditions (33) becomes

$$\begin{aligned} \alpha(t) &= \frac{1}{\cosh(\Omega_N T)} \left\{ \alpha' e^{-i\omega_m t} \cosh[\Omega_N(T-t)] - i\eta'' e^{i\omega_m(T-t)} \sinh(\Omega_N t) \right\} \\ \beta(t) &= \frac{1}{\cosh(\Omega_N T)} \left\{ \beta'' e^{-i\omega_m(T-t)} \cosh(\Omega_N t) - i\zeta' e^{i\omega_m t} \sinh[\Omega_N(T-t)] \right\} \\ \delta\zeta(t) &= \frac{1}{\cosh(\Omega_N T)} \left\{ \zeta' e^{i\omega_m t} \cosh[\Omega_N(T-t)] - i\beta'' e^{-i\omega_m(T-t)} \sinh(\Omega_N t) \right\} \\ \delta\eta(t) &= \frac{1}{\cosh(\Omega_N T)} \left\{ \eta'' e^{i\omega_m(T-t)} \cosh(\Omega_N t) - i\alpha' e^{-i\omega_m t} \sinh[\Omega_N(T-t)] \right\} \end{aligned} \quad (52)$$

with the frequencies

$$\begin{aligned} \omega_m &= 1 + \frac{\Delta}{2} \\ \Omega_N &= \sqrt{\lambda^2 - \frac{\Delta^2}{4}}. \end{aligned} \quad (53)$$

Note that for long times the dominant path converges to the corresponding boundary value and no oscillations around the north pole take place anymore.

In the same way, we linearize the motion around the south pole. Now invariant subspaces appear in the variables  $(\delta\rho, \alpha)$  and  $(\delta\sigma, \beta)$

$$\frac{d}{dt} \begin{pmatrix} \delta\rho \\ \alpha \\ \delta\sigma \\ \beta \end{pmatrix} = i \begin{pmatrix} -1 - \Delta & -\lambda & 0 & 0 \\ -\lambda & -1 & 0 & 0 \\ 0 & 0 & 1 + \Delta & \lambda \\ 0 & 0 & \lambda & 1 \end{pmatrix} \begin{pmatrix} \delta\rho \\ \alpha \\ \delta\sigma \\ \beta \end{pmatrix}, \quad (54)$$

with the solution

$$\begin{aligned} \alpha(t) &= \alpha' e^{-i\omega_m t} \cos(\Omega_S t) - i \frac{1}{\zeta'} e^{-i\omega_m t} \sin(\Omega_S t) \\ \beta(t) &= \beta'' e^{-i\omega_m(T-t)} \cos[\Omega_S(T-t)] - i \frac{1}{\eta''} e^{-i\omega_m(T-t)} \sin[\Omega_S(T-t)] \\ \delta\rho(t) &= \frac{1}{\zeta'} e^{-i\omega_m t} \cos(\Omega_S t) - i \alpha' e^{-i\omega_m t} \sin(\Omega_S t) \\ \delta\sigma(t) &= \frac{1}{\eta''} e^{-i\omega_m(T-t)} \cos[\Omega_S(T-t)] - i \beta'' e^{-i\omega_m(T-t)} \sin[\Omega_S(T-t)], \end{aligned} \quad (55)$$

where

$$\Omega_S = \sqrt{\lambda^2 + \frac{\Delta^2}{4}}. \quad (56)$$

Here, the dominant path does not converge for long times but keeps on oscillating around the south pole. North pole and south pole correspond to the local extrema of the cubic potential (37) generated by the coupling of the spin- $\frac{1}{2}$  to a vacuum field. Whenever the field becomes filled with bosons, these fix points bifurcate into limit cycles.

The presence of stationary states leads to strong deviations of the DOPA propagator from the exact result for times large compared to  $\omega_o^{-1}$ . In fact, for long times the semiclassical trajectory approaches the saddle point of the cubic potential and stays there for most of the time. For the full quantum problem the state  $|\uparrow 0\rangle$  is not a steady state, rather it will decay by spontaneous emission. In the semiclassical approximation spontaneous emission arises from fluctuations about the classical path that are neglected in the DOPA. Hence, to obtain useful results also for long times, fluctuations about the north pole need to be taken into account.

#### IV. FLUCTUATIONS

The semiclassical expansion of the path integral (8) leads to second order contributions in terms of Gaussian fluctuation path integrals. Denoting by  $(x_a, y_a)$  and  $(x_b, y_b)$  deviations from the dominant path variables  $(p, q)$  and  $(\cos(\vartheta), \varphi)$ , the semiclassical approximation takes the form

$$\langle \vartheta'' \varphi'' p'' q'' | U(T) | \vartheta' \varphi' p' q' \rangle_{sc} = e^{iS_{cl}} \lim_{\nu_a, \nu_b \rightarrow \infty} \int d\mu_a d\mu_b \exp \left\{ i \delta^2 S[x_a(t), y_a(t), x_b(t), y_b(t)] \right\}, \quad (57)$$

with the boundary conditions  $x_a(0) = x_a(T) = 0$ ,  $y_a(0) = y_a(T) = 0$ ,  $x_b(0) = x_b(T) = 0$ ,  $y_b(0) = y_b(T) = 0$ . Since the canonical Wiener measure (25) is of quadratic form, the measure of the fluctuation path integral becomes

$$d\mu_a = \prod_{t=0}^T \frac{1}{2\pi} dx_a(t) dy_a(t) \exp \left\{ -\frac{1}{2\nu_a} \int_0^T dt [\dot{x}_a^2 + \dot{y}_a^2] \right\}, \quad (58)$$

which is of the same form as the original coherent state path measure. On the other hand, the spin measure (26) is not quadratic, and the dominant path  $(\vartheta(t), \varphi(t))$  cannot be separated from the fluctuation variables  $x_b$  and  $y_b$ . We have

$$d\mu_b = \prod_{t=0}^T \frac{2s+1}{4\pi s} dx_b(t) dy_b(t) \exp \left\{ -\frac{1}{2\nu_b} \int_0^T dt \left[ \frac{\dot{x}_b^2}{\sin^2(\vartheta)} + \sin^2(\vartheta) \dot{y}_b^2 \right. \right. \\ \left. \left. - \cos(\vartheta) \dot{\varphi} x_b \dot{y}_b + 2 \frac{\cos(\vartheta) \dot{\vartheta}}{\sin^3(\vartheta)} \dot{x}_b x_b + \left( \dot{\varphi}^2 - \frac{2 \cos^2(\vartheta) \dot{\vartheta}^2 + \sin(2\vartheta) \ddot{\vartheta}}{2 \sin^4(\vartheta)} \right) x_b^2 \right] \right\}, \quad (59)$$

and the regularization of the fluctuation path integral becomes in general time dependent. However, when the dominant spin path is strictly independent of time,  $(\vartheta(t), \varphi(t)) = (\vartheta_o, \varphi_o)$ , the measure (59) simplifies considerably and we get

$$d\mu_b = \prod_{t=0}^T \frac{2s+1}{4\pi s} dx_b(t) dy_b(t) \exp \left\{ -\frac{1}{2\nu_b} \int_0^T dt \left[ \frac{\dot{x}_b^2}{\sin^2(\vartheta_o)} + \sin^2(\vartheta_o) \dot{y}_b^2 \right] \right\}. \quad (60)$$

Then, after a canonical transformation

$$\begin{aligned} \tilde{x}_b &= \frac{x_b}{\sin(\vartheta_o)} \\ \tilde{y}_b &= \sin(\vartheta_o) y_b, \end{aligned} \quad (61)$$

the measure(60) takes for large  $s$  the form of the canonical measure (58)

$$d\mu_b = \int \prod_{t=0}^T \frac{1}{2\pi} d\tilde{x}_b(t) d\tilde{y}_b(t) \exp \left\{ -\frac{1}{2\nu_b} \int_0^T dt [\dot{\tilde{x}}_b^2 + \dot{\tilde{y}}_b^2] \right\}. \quad (62)$$

Both measures give rise to the regularization dependent second order variational action

$$\begin{aligned} \delta^2 S_{\nu_a, \nu_b} [x_a(t), y_a(t), \tilde{x}_b(t), \tilde{y}_b(t)] &= \int_0^T dt \left[ \frac{i}{2\nu_a} (\dot{x}_a^2 + \dot{y}_a^2) + \frac{i}{2\nu_b} (\dot{\tilde{x}}_b^2 + \dot{\tilde{y}}_b^2) \right. \\ &\quad \left. + \frac{1}{2} (x_a \dot{y}_a - \dot{x}_a y_a) + \frac{1}{2} (\tilde{x}_b \dot{\tilde{y}}_b - \dot{\tilde{x}}_b \tilde{y}_b) - H_o(x_a, y_a, \tilde{x}_b, \tilde{y}_b, t) \right], \end{aligned} \quad (63)$$

where the Hamiltonian  $H_o(t)$  is determined by the second order contributions of the Hamiltonian  $H$  expanded around the dominant path

$$\begin{aligned} H_o(x_a, y_a, \tilde{x}_b, \tilde{y}_b, t) &= a_1(t) x_a^2 + a_2(t) x_a y_a + a_3(t) y_a^2 \\ &\quad + b_1(t) \tilde{x}_b^2 + b_2(t) \tilde{x}_b \tilde{y}_b + b_3(t) \tilde{y}_b^2 \\ &\quad + c_1(t) x_a \tilde{x}_b + c_2(t) x_a \tilde{y}_b + c_3(t) y_a \tilde{x}_b + c_4(t) y_a \tilde{y}_b, \end{aligned} \quad (64)$$

with the coefficients

$$\begin{aligned}
a_1(t) &= \frac{1}{2} \frac{\partial^2 H}{\partial \bar{q}^2}, & a_2(t) &= \frac{\partial^2 H}{\partial \bar{q} \partial \bar{p}}, & a_3(t) &= \frac{1}{2} \frac{\partial^2 H}{\partial \bar{p}^2} \\
b_1(t) &= \frac{\sin^2(\vartheta_o)}{2s} \frac{\partial^2 H}{\partial \cos(\bar{\vartheta})^2}, & b_2(t) &= \frac{1}{s} \frac{\partial^2 H}{\partial \bar{\varphi} \partial \cos(\bar{\vartheta})}, & b_3(t) &= \frac{1}{2s \sin^2(\vartheta_o)} \frac{\partial^2 H}{\partial \bar{\varphi}^2} \\
c_1(t) &= \frac{\sin(\vartheta_o)}{\sqrt{s}} \frac{\partial^2 H}{\partial \bar{p} \partial \cos(\bar{\vartheta})}, & c_2(t) &= \frac{1}{\sqrt{s} \sin(\vartheta_o)} \frac{\partial^2 H}{\partial \bar{p} \partial \bar{\varphi}}, \\
c_3(t) &= \frac{\sin(\vartheta_o)}{\sqrt{s}} \frac{\partial^2 H}{\partial \bar{q} \partial \cos(\bar{\vartheta})}, & c_4(t) &= \frac{1}{\sqrt{s} \sin(\vartheta_o)} \frac{\partial^2 H}{\partial \bar{q} \partial \bar{\varphi}}.
\end{aligned} \tag{65}$$

For large  $s$ , starting and end points in the coherent state fluctuation path integral (57) parameterize states  $|x_a(0) y_a(0) x_b(0) y_b(0)\rangle$  and  $|x_a(T) y_a(T) x_b(T) y_b(T)\rangle$  which correspond to product vacuum states. Propagators leading to stationary saddle points  $(\vartheta_o, \varphi_o)$  may be represented now as

$$\langle \vartheta_o \varphi_o p'' q'' | U(T) | \vartheta_o \varphi_o p' q' \rangle_{sc} = e^{iS_{cl}} \langle 00 | U_o(T) | 00 \rangle, \tag{66}$$

with the unitary time evolution operator

$$U_o(T) = \mathcal{T}_t \exp \left\{ -i \int_0^T dt H_o(t) \right\}, \tag{67}$$

determined by the quadratic Hamiltonian

$$\begin{aligned}
H_o(t) &= a_1(t) \left( Q_a^2 - \frac{1}{2} \right) + a_2(t) (P_a Q_a + Q_a P_a) + a_3(t) \left( P_a^2 - \frac{1}{2} \right) \\
&\quad + b_1(t) \left( Q_b^2 - \frac{1}{2} \right) + b_2(t) (P_b Q_b + Q_b P_b) + b_3(t) \left( P_b^2 - \frac{1}{2} \right) \\
&\quad + c_1(t) P_a P_b + c_2(t) P_a Q_b + c_3(t) Q_a P_b + c_4(t) Q_a Q_b,
\end{aligned} \tag{68}$$

describing two driven coupled oscillators.

As we have seen in the previous section, for the Jaynes-Cummmings model the north pole  $|\uparrow 0\rangle$  becomes a steady state in the DOPA, and it is essential to take fluctuations about this state into account. Unfortunately, the description of the spin degrees of freedom with spherical coordinates leads to coordinate singularities. Particularly, the azimuthal angle  $\varphi$  is undefined at the poles of the two-sphere. To calculate fluctuations about the north pole accurately, we change the coordinate system by a rotation. Since rotations are isometrical canonical transformations, the spin path measure (26) stays invariant but the kinematical term is not preserved. Instead a phase factor appears which only vanishes if starting and endpoint of the spin coordinates are identical.

Within the DOPA, the probability amplitude to remain at the north pole is just a phase factor

$$\langle \uparrow 0 | U(T) | \uparrow 0 \rangle_{DOPA} = e^{iS_{cl}} = \exp \left\{ -i \frac{1 + \Delta}{2} T \right\}, \tag{69}$$

and the north pole becomes a steady state. Taking now Gaussian fluctuations into account we have

$$\langle \uparrow 0 | U(T) | \uparrow 0 \rangle_{sc} = e^{iS_{cl}} \langle 0 0 | U_o(T) | 0 0 \rangle, \quad (70)$$

where the the vacuum amplitude is determined by the time independent Hamiltonian

$$H_o = \frac{1}{2} (P_a^2 + Q_a^2 - 1) - \frac{1 + \Delta}{2} (P_b^2 + Q_b^2 - 1) + \lambda (P_b Q_a + Q_b P_a). \quad (71)$$

For convenience we represent the operators  $Q_a, P_a$  and  $Q_b, P_b$  by corresponding creation and annihilation operators  $a, a^\dagger$  and  $b, b^\dagger$

$$H_o = -1 + (1 + \frac{\Delta}{2})(aa^\dagger - b^\dagger b) - \frac{\Delta}{2}(aa^\dagger + b^\dagger b) - i\lambda(ab - a^\dagger b^\dagger). \quad (72)$$

Since  $aa^\dagger - b^\dagger b$  commutes with  $H_o$ , we rewrite the time evolution operator in the form

$$U_o(T) = \exp\left\{-i\frac{\Delta T}{2}\right\} \exp\left\{-i(1 + \frac{\Delta}{2})a^\dagger a T\right\} \exp\left\{i(1 + \frac{\Delta}{2})b^\dagger b T\right\} U_1(T), \quad (73)$$

with  $U_1(T) = \exp(-iH_1 T)$  and

$$H_1 = -\left[\frac{\Delta}{2}(aa^\dagger + b^\dagger b) + i\lambda(ab - a^\dagger b^\dagger)\right]. \quad (74)$$

The operators  $aa^\dagger + b^\dagger b$ ,  $ab$  and  $a^\dagger b^\dagger$  span the three dimensional  $su(1, 1)$  Lie algebra with commutators

$$\begin{aligned} [aa^\dagger + b^\dagger b, ab] &= -2ab \\ [aa^\dagger + b^\dagger b, a^\dagger b^\dagger] &= 2a^\dagger b^\dagger \\ [ab, a^\dagger b^\dagger] &= aa^\dagger + b^\dagger b. \end{aligned} \quad (75)$$

For this algebra there is a decomposition into one-dimensional  $SU(1, 1)$  transformations which holds for the whole group, i.e. for all times [22]. We start with the ansatz

$$U_1(T) = \exp\{\mu(T)a^\dagger b^\dagger\} \exp\{\nu(T)ab\} \exp\{\xi(T)(aa^\dagger + b^\dagger b)\}, \quad (76)$$

which results in the vacuum amplitude

$$\langle 0 0 | U_o(T) | 0 0 \rangle = \exp\left\{-i\frac{\Delta}{2}T + \xi(T)\right\}. \quad (77)$$

Then, requiring that  $U_1(T)$  obeys the Schrödinger equation  $d/dT U_1(T) = -iH_1 U_1(T)$ , we get the relation

$$\begin{aligned} i\frac{\Delta}{2}(aa^\dagger + b^\dagger b) + \lambda(a^\dagger b^\dagger - ab) &= \dot{\mu}a^\dagger b^\dagger + \dot{\nu}e^{\mu a^\dagger b^\dagger} ab e^{-\mu a^\dagger b^\dagger} \\ &\quad + \dot{\xi}e^{\mu a^\dagger b^\dagger} e^{\nu ab}(aa^\dagger + b^\dagger b)e^{-\nu ab}e^{-\mu a^\dagger b^\dagger}, \end{aligned} \quad (78)$$

where we have made use of the Baker-Campbell-Hausdorff formula. Further, the commutation relations (75) imply

$$\begin{aligned}
e^{\nu ab}(aa^\dagger + b^\dagger b)e^{-\nu ab} &= aa^\dagger + b^\dagger b + 2\nu ab \\
e^{\mu a^\dagger b^\dagger}(aa^\dagger + b^\dagger b)e^{-\mu a^\dagger b^\dagger} &= aa^\dagger + b^\dagger b - 2\mu a^\dagger b^\dagger \\
e^{\mu a^\dagger b^\dagger}ab e^{-\mu a^\dagger b^\dagger} &= ab - \mu(aa^\dagger + b^\dagger b) + \mu^2 a^\dagger b^\dagger.
\end{aligned} \tag{79}$$

Now, Eq.(78) determines the time rate of change of the functions  $\mu$ ,  $\nu$  and  $\xi$  by the linear equations

$$\begin{pmatrix} \dot{\lambda} \\ \dot{\lambda} \\ i\dot{\Delta} \end{pmatrix} = \begin{pmatrix} 1 & \mu^2 & -2\mu(1 - \mu\nu) \\ 0 & -1 & -2\nu \\ 0 & -2\mu & 2(1 - 2\mu\nu) \end{pmatrix} \begin{pmatrix} \dot{\mu} \\ \dot{\nu} \\ \dot{\xi} \end{pmatrix}, \tag{80}$$

which are readily solved with the initial conditions  $\mu(0) = 0$ ,  $\nu(0) = 0$  and  $\xi(0) = 0$ . In particular, we get for the function  $\xi(T)$  in Eq.(76)

$$\xi(T) = i\Delta T + \log \left[ \cos(\Omega T) - i\frac{\Delta}{2} \sin(\Omega T) \right], \tag{81}$$

with the Rabi frequency

$$\Omega = \sqrt{\lambda^2 + \frac{\Delta^2}{4}}. \tag{82}$$

Hence, the vacuum amplitude (77) becomes

$$\langle 00|U_o(T)|00\rangle = \exp\left\{i\frac{\Delta T}{2}\right\} \left[ \cos(\Omega T) - i\frac{\Delta}{2} \sin(\Omega T) \right]. \tag{83}$$

and the semiclassical propagator with fluctuations

$$\langle \uparrow 0|U(T)|\uparrow 0\rangle_{sc} = e^{-\frac{i}{2}T} \left[ \cos(\Omega T) - i\frac{\Delta}{2} \sin(\Omega T) \right] \tag{84}$$

includes spontaneous emission leading to an instability of the north pole. Eq.(84) gives the exact matrix element of the propagator sandwiched between north pole states.

When the field is initially and finally not in the vacuum state, the semiclassical propagator (57) is no longer characterized by a fix point path. An evaluation of the fluctuations about the semiclassical path would then require numerical methods beyond the scope of this article.

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## APPENDIX A: SEMICLASSICAL SCHRÖDINGER EQUATION

Here we derive the semiclassical Schrödinger equation for the DOPA propagator given in Eq.(30). The time rate of change is readily evaluated, and after an integration by parts it may be expressed as

$$\begin{aligned}
\frac{\partial}{\partial T} e^{iS_{cl}} = & \frac{1}{2} \left\{ -\frac{\partial \alpha(T, T)}{\partial T} \beta'' - \alpha' \frac{\partial \beta(0, T)}{\partial T} - \frac{\zeta' \frac{\partial \eta(0, T)}{\partial T}}{1 + \zeta' \eta(0, T)} - \frac{\frac{\partial \zeta(T, T)}{\partial T} \eta''}{1 + \zeta(T, T) \eta''} \right. \\
& - \frac{\partial \alpha(t, T)}{\partial t} \Big|_T \beta'' + \alpha(T, T) \frac{\partial \beta(t, T)}{\partial t} \Big|_T - \frac{\frac{\partial \zeta(t, T)}{\partial t} \Big|_T \eta'' - \zeta(T, T) \frac{\partial \eta(t, T)}{\partial t} \Big|_T}{1 + \zeta(T, T) \eta''} \\
& - 2iH(\alpha(T, T), \beta'', \zeta(T, T), \eta'') \\
& - \left[ \frac{\partial \alpha(t, T)}{\partial T} \beta(t, T) - \alpha(t, T) \frac{\partial \beta(t, T)}{\partial T} + \frac{\frac{\partial \zeta(t, T)}{\partial T} \eta(t, T) - \zeta(t, T) \frac{\partial \eta(t, T)}{\partial T}}{1 + \zeta(T, T) \eta''} \right]_{t=0}^{t=T} \\
& - \int_0^T dt \left[ \frac{\alpha(t, T)}{\partial T} \left( \frac{\partial \beta(t, T)}{\partial t} - i \frac{\partial H}{\partial \alpha} \right) - \frac{\beta(t, T)}{\partial T} \left( \frac{\partial \alpha(t, T)}{\partial t} + i \frac{\partial H}{\partial \beta} \right) \right. \\
& \quad \frac{\zeta(t, T)}{\partial T} \left( \frac{\frac{\partial \eta(t, T)}{\partial t}}{(1 + \zeta(t, T) \eta(t, T))^2} - i \frac{\partial H}{\partial \zeta} \right) \\
& \quad \left. \left. - \frac{\eta(t, T)}{\partial T} \left( \frac{\frac{\partial \zeta(t, T)}{\partial t}}{(1 + \zeta(t, T) \eta(t, T))^2} + i \frac{\partial H}{\partial \eta} \right) \right] \right\} e^{iS_{cl}}. \tag{A1}
\end{aligned}$$

(A2)

Using the classical equations of motions, the integral is found to vanish. Then, we rewrite the remaining parts in the form

$$\begin{aligned}
\frac{\partial}{\partial T} e^{iS_{cl}} = & -iH(\alpha(T, T), \beta'', \zeta(T, T), \eta'') \\
& - \frac{1}{2} \left[ \beta'' \left( -\frac{\partial \alpha(T, T)}{\partial T} + \frac{\partial \alpha(t, T)}{\partial t} \Big|_T + \frac{\partial \alpha(t, T)}{\partial T} \Big|_T \right) \right. \\
& + \alpha' \left( -\frac{\partial \beta(0, T)}{\partial T} + \frac{\partial \beta(t, T)}{\partial T} \Big|_0 \right) \\
& + \alpha(T, T) \left( -\frac{\partial \beta(t, T)}{\partial t} \Big|_T - \frac{\partial \beta(t, T)}{\partial T} \Big|_T \right) - \beta(0, T) \frac{\partial \alpha(t, T)}{\partial T} \Big|_0 \\
& + \frac{\eta'' \left( -\frac{\partial \zeta(T, T)}{\partial T} + \frac{\partial \zeta(t, T)}{\partial t} \Big|_T + \frac{\partial \zeta(t, T)}{\partial T} \Big|_T \right) + \zeta(T, T) \left( -\frac{\partial \eta(t, T)}{\partial t} \Big|_T - \frac{\partial \eta(t, T)}{\partial T} \Big|_T \right)}{1 + \zeta(T, T) \eta''} \\
& \left. + \frac{\zeta' \left( -\frac{\partial \eta(0, T)}{\partial T} + \frac{\partial \eta(t, T)}{\partial t} \Big|_0 \right) - \eta(0, T) \frac{\partial \zeta(0, T)}{\partial T} \Big|_0}{1 + \zeta' \eta(0, T)} \right] e^{iS_{cl}}, \tag{A3}
\end{aligned}$$

where most of the terms on the right hand side vanish. Finally we get

$$\frac{\partial}{\partial T} e^{iS_{cl}} = -iH(\alpha(T), \beta'', \zeta(T), \eta'') e^{iS_{cl}}. \tag{A4}$$

Note that the matrix element of the Hamiltonian at the endpoint of the dominant path  $(\alpha(T), \beta'', \zeta(T), \eta'')$  generates the time rate of change of the DOPA propagator and not the matrix element of the final state  $|\vartheta'' \varphi'' p'' q''\rangle$ . For a spin- $\frac{1}{2}$  coupled to a classical field this Schrödinger equation generates the exact quantum mechanics [11].

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